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A new model of the Calogero type

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Abstract

We propose a new integrable Hamiltonian describing two interacting particles in a harmonic mean field in $D = 1$ dimensional space. This model is found to be both supersymmetric and shape invariant. We show that for a given domain of the coupling constant the irregular solution is acceptable and contributes to the spectrum. We also discuss two inequalities of the Bertlmann–Martin type, which link the ground state mean-square radius to the lowest excitation energy.

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1. Introduction

In the three-body problem, exactly solvable models are scarce. Among the most famous, the Calogero model has attracted a lot of attention [1], giving rise to numerous works and extensions. Integrable systems of this kind have been reviewed and classified by Olshanetsky and Perelomov [2]. The purpose of the present work is to show a new integrable model, which, though slightly different, is related to cases of the class V of Olshanetsky and Perelomov. These authors stress the fact that the integrability results from the presence of (hidden) symmetries. Indeed, we shall show that the model proposed here is supersymmetric and shape invariant.

The paper is organized as follows. In section 2 the model is developed and discussed. Section 3 is devoted to the supersymmetry and in section 4 the Bertlmann–Martin inequality is tested. Conclusions are drawn in section 5.

2. The model

In the $D = 1$ dimensional space, we consider two interacting particles which are bound within a harmonic mean field. The Hamiltonian that we shall study is

$$H(x_1, x_2) = -\frac{d^2}{dx_1^2} - \frac{d^2}{dx_2^2} + \omega^2\left(x_1^2 + x_2^2\right) + \lambda\frac{2x_1x_2}{\left(x_1^2 + x_2^2\right)(x_1 - x_2)^2}, \tag{1}$$

where we use units $2m = \hbar = 1$. This Hamiltonian is invariant under the $x_1 \rightleftharpoons x_2$ exchange. It describes systems where there is one very heavy particle that generates a mean field which is experienced by two light particles that are mutually interacting. It has some similarities to the 'simple' Calogero model, which, by using the Jacobi coordinates, reads

$$H(x_1, x_2) = -\frac{\mathrm{d}^2}{\mathrm{d}x_1^2} - \frac{\mathrm{d}^2}{\mathrm{d}x_2^2} + \omega^2\left(x_1^2 + x_2^2\right) + \frac{g}{x_1^2}. \tag{2}$$

Whereas the Hamiltonian (2) is separable in the $\{x_1, x_2\}$ coordinates, that of (1) is not. However, this drawback can be overcome by introducing polar coordinates

$$x_1 = r\sin\varphi, \qquad x_2 = r\cos\varphi \qquad r \geqslant 0. \tag{3}$$

The Hamiltonian (1) then reads

$$-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \omega^2 r^2 + \frac{\lambda}{r^2}\frac{\sin 2\varphi}{1 - \sin 2\varphi}. \tag{4}$$

Actually, the Hamiltonian (1) belongs to the class of conformal quantum many-body systems studied by Meljanac and Samsarov [3].

We look for eigenenergies $E_{k,n}$ of the above operator, where k and n label the radial and angular states, respectively. This requires the solving of a two-variables differential equation, which can be separated in the form

$$\Psi_{k,n}(r,\varphi) = u_k(r)\phi_n(\varphi). \tag{5}$$

We then obtain the two differential equations

$$\left[\frac{\partial^2}{\partial\varphi^2} - \lambda\frac{\sin 2\varphi}{1 - \sin 2\varphi} + C_n\right]\phi_n(\varphi) = 0, \tag{6}$$

and

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \omega^2 r^2 + E_{k,n} - \frac{C_n}{r^2}\right]u_k(r) = 0. \tag{7}$$

A first remark concerns the domain of validity of equation (6). Being a function of $\sin(2\varphi)$, the potential has a periodicity of π. There is a singularity for $\varphi = \pi/4 + k\pi$, so we choose to study the interval $\varphi \in [\pi/4, \pi/4 + \pi]$.

Setting $\varphi = \pi/4 + \theta$ with $\theta \in [0, \pi]$, equation (6) is rewritten in terms of θ as

$$\left[\frac{\partial^2}{\partial\theta^2} - \lambda\frac{\cos 2\theta}{1 - \cos 2\theta} + C_n\right]\phi_n(\theta) = 0, \tag{8}$$

and solved in the interval $[0, \pi]$.

Note, this equation is essentially the one-body inverse-sine squared problem of Sutherland [4].

Since $1 - \cos(2\theta)$ behaves like $2(\theta)^2$ in the vicinity of 0 and π, the singularity is analogous to that of a centrifugal barrier. It can be treated if and only if $1/4 + \lambda/2 \geqslant 0$, or equivalently $\lambda \geqslant -1/2$ [5]. Note that for $\lambda = -1/2$ the operator has several self-adjoint extensions.

We remark that for $\lambda = 0$, the barrier disappears and the solution extends over the whole $[0, 2\pi]$ interval, and in this way, we recover the usual harmonic oscillator in $D = 2$. However, there is no continuity between the two regimes since the barrier exists for any non-zero λ. Trying to express the solutions of equation (8) on the interval $[0, \pi]$ with Dirichlet conditions at the boundaries in terms of orthogonal polynomials, we set

$$\begin{aligned} &\phi_n(\theta) = (\sin\theta)^{2m} g_n(y) \\ &y = \cos\theta. \end{aligned} \tag{9}$$

Equation (8) is then transformed to

$$\left[(1 - y^2)\frac{\partial^2}{\partial y^2} - (4m + 1)y\frac{\partial}{\partial y} + C_n + 4m(m - 1)\right]g_n(y) = 0, \tag{10}$$

where

$$\lambda = 4m(2m-1), \tag{11}$$

and thus

$$m = \frac{1 \pm \sqrt{1+2\lambda}}{4}. \tag{12}$$

The latter equation has real solutions m if and only if $\lambda \geqslant -1/2$, which is the condition for the existence of acceptable solutions to equation (6) near the singularity. Generally, only the regular solutions are retained. The irregular solution is usually excluded on the grounds that the corresponding eigensolutions are not squared integrable. In certain cases, the exclusion is motivated by a pathology as, for instance, the occurrence of a δ distribution when the Laplacian is applied to the solution. However, as shown by Murthy *et al* [6], square integrable irregular solutions may exist. Their proof is given in a context similar to the present one, viz anyons confined in a harmonic oscillator. Consequently, we include in our discussion the irregular solutions and the corresponding spectrum.

The general solutions of equation (10) are Jacobi polynomials $P_n^{\alpha,\alpha}(y)$ with $\alpha = (4m-1)/2$ and $C_n = n(n+4m) - 4m(m-1)$.

From equation (7) on the interval $[0, +\infty[$ we have the requirement that

$$C_n = n(n+4m) - 4m(m-1) = (n+2m)^2 - \lambda \geqslant 0, \qquad \forall\, n. \tag{13}$$

The squared term is minimal when $n = 0$. Combined with equation (11) this implies that $m \in [0, 1]$ so that $\lambda \in [-1/2, 4]$. Under this constraint, C_n is positive for every n. Guided by all these information we conclude that the general solution of equation (8) is of the form

$$\phi_n(\theta) = (\sin\theta)^{2m} P_n^{\alpha,\alpha}(\cos\theta) \tag{14}$$

with $m \in [0, 1]$, $\alpha = (4m-1)/2$.

The square integrability of ϕ_n on $[0, \pi]$ is ensured for $m \geqslant 0$ due to the positive root in equation (12). However, for $\lambda \in [-1/2, 0]$, the negative root also leads to positive m. Therefore, on this domain of λ, both solutions of equation (12) have to be considered. We are thus facing a situation in which the regular and the irregular solutions are both acceptable [6]. The irregular solution is defined here as the one corresponding to the lowest value of m in equation (12) following the definition of [5] for the centrifugal barrier.

For $\lambda \in [0, 4]$, only one positive solution $m \in [0, 1]$ is found. For $\lambda = -1/2$ both solutions equation (12) coincide and we take the general solution of equation (14) for $m = 1/4$ by continuity argument.

In terms of the variable φ we have (modulo the multiplicative constant 2^m)

$$\phi_n(\varphi) = (1 - \sin(2\varphi))^m P_n^{\alpha,\alpha}\left(\frac{\sin(\varphi) + \cos(\varphi)}{\sqrt{2}}\right). \tag{15}$$

For $m \neq 0$ the solution ϕ_n vanishes at the boundaries $\varphi = \pi/4, \pi/4 + \pi$.

As far as the radial equation is concerned, we first introduce $u_k(r) = v_k(r)/\sqrt{r}$ to yield the reduced radial equation

$$\left[\frac{\partial^2}{\partial r^2} - \omega^2 r^2 - \frac{C_n - 1/4}{r^2} + E\right] v_k(r) = 0. \tag{16}$$

We further introduce the auxiliary quantity ℓ, defined by

$$C_n = \ell^2 \qquad \ell = \sqrt{n(n+4m) - 4m(m-1)}, \tag{17}$$

(the negative values of ℓ have been disregarded as leading for $n \geqslant 1$ to non-square-integrable eigensolutions) and consider the particular solutions

$$v_k(r) = r^{\ell+1/2} \exp(-\omega r^2/2) f_k(\omega r^2). \tag{18}$$

Inserting this ansatz into the reduced radial equation, we obtain the differential equation for f_k

$$z f_k''(z) + (\ell + 1 - z) f_k'(z) + \left(\frac{E}{4\omega} - \frac{\ell}{2} - \frac{1}{2}\right) f_k(z) = 0 \tag{19}$$

where $z = \omega r^2$ and the prime denotes the derivative with respect to z. Equation (19) is nothing but the Laguerre polynomial differential equation with solutions

$$f_k(z) = L_k^{\ell}(z) \qquad k = 0, 1, 2, .. \tag{20}$$

Putting these various elements together, the solutions of the total Hamiltonian are expressed by the wavefunctions

$$\Psi_{k,n}(r,\varphi) = r^{\ell} \exp\left(-\frac{\omega}{2} r^2\right) L_k^{\ell}(\omega r^2)(1 - \sin(2\varphi))^m P_n^{\alpha,\alpha}\left(\frac{\sin(\varphi) + \cos(\varphi)}{\sqrt{2}}\right). \tag{21}$$

The associated eigenvalues are given by

$$E_{k,n} = 2\omega(2k + \sqrt{n(n+4m) - 4m(m-1)} + 1), \tag{22}$$

or equivalently

$$E_{k,n} = 2\omega(2k + \sqrt{(n+2m)^2 - \lambda} + 1). \tag{23}$$

We recall that the indices $\{k, n\}$ denote the azimuthal quantum number and the number of nodes of the wavefunction, respectively. The link with the ordinary harmonic oscillator in 2D is made by observing that for $m = 0$ and $m = 1/2$ the coupling constant λ vanishes and we have $\ell = n$ for $\lambda = 0$ and $\ell = n + 1$ for $\lambda = 1/2$.

Expressed in terms of x_1, x_2 the eigensolutions read

$$\Psi_{k,n}(x_1,x_2) = \left(x_1^2 + x_2^2\right)^{\ell/2 - m} \exp\left(-\frac{\omega}{2}\left(x_1^2 + x_2^2\right)\right) L_k^{\ell}\left(\omega\left(x_1^2 + x_2^2\right)\right) \\ \times (x_1 - x_2)^{2m} P_n^{\alpha,\alpha}\left(\frac{x_1 + x_2}{\sqrt{2\left(x_1^2 + x_2^2\right)}}\right). \tag{24}$$

Since the solutions are taken on $\theta \in [0, \pi]$, we have $x_1 - x_2 = \sqrt{2} r \sin\theta \geqslant 0$, which allows us to define $(x_1 - x_2)^{2m}$ for real values of m in the interval [0, 1].

The eigensolution equations (21) are orthogonal for $\theta \in [0, \pi]$ in the sense that

$$\int_0^{+\infty} r\,\mathrm{d}r \int_0^{\pi} \mathrm{d}\theta\ \Psi_{k,n}(r,\varphi)\Psi_{k',n'}(r,\varphi) = \delta_{n,n'}\delta_{k,k'} N_{k,n} \tag{25}$$

with $\varphi = \pi/4 + \theta$. The normalization constant $N_{k,n}$ is given by

$$N_{k,n} = 4^{3m-1} \frac{\Gamma(\ell + 1 + k)\Gamma(n + (4m+1)/2)^2}{k!n!(n+2m)\Gamma(n+4m)} w^{-\ell-1} \qquad n + m > 0 \\ = \frac{\pi}{2} w^{-1} \qquad n = m = 0. \tag{26}$$

The separability of the variables and the fact that the solutions are expressed by Laguerre and Jacobi polynomials, respectively, ensure that the normalized solutions equation (21) constitutes a complete orthonormal basis [7].

We end this section with a short discussion of the spectrum given by equation (23). For $\lambda \in [-1/2, 0]$, both values of m are acceptable. It leads to

$$E_{k,n}^{>,<} = 2\omega\left(2k + \sqrt{(n + 2m_{>,<})^2 - \lambda} + 1\right). \tag{27}$$

Here the symbols >, < denote respectively the upper and lower value of m. For fixed λ the energies of the irregular solutions, corresponding to $m_<$, are lower than the regular ones.

Let us consider the limiting cases $m = 0$ and $m = 1/2$. The value of $m = 0$ leads to the usual spectrum

$$E_{k,n} = 2\omega(2k + n + 1). \tag{28}$$

The eigensolutions are proportional to $\cos(n\theta)$. For $m = 1/2$, we have

$$E_{k,n} = 2\omega(2k + n + 2). \tag{29}$$

The spectrum is shifted and the eigensolutions are proportional to $\sin(n\theta)$. More explicitly, the ground state energy for $m = 1/2$ (equation (29) for $n = 0$) is the same as the first excited state energy for $m = 0$ (equation (28) for $n = 1$).

Note that both solutions $\{\cos(n\theta), \sin(n\theta)\}$, where n is a positive integer, can be extended naturally to $[-\pi, \pi]$ and the above set leads to the usual harmonics $\exp(\mathrm{i}p\theta)$, with integer p, on $[0, 2\pi]$.

3. Supersymmetry

The question is to examine in what respect equation (6) has supersymmetric shape invariant partners. The basic elements used in this section are well known and we refer the reader to the comprehensive paper by Dutt *et al* [8].

We introduce

$$W(\varphi) = 2m\frac{\cos 2\varphi}{1 - \sin 2\varphi} \tag{30}$$

and consider

$$V_\pm = W^2 \pm \frac{\mathrm{d}}{\mathrm{d}\varphi}W. \tag{31}$$

This leads to

$$V_\pm(\varphi) = 4m(m \pm 1) + (8m^2 \pm 4m)\frac{\sin 2\varphi}{1 - \sin 2\varphi}, \tag{32}$$

which is shape invariant.

As far as the radial part is concerned, it is known to be supersymmetric and shape invariant [8] but, for the sake of completeness, we sketch the proof. Starting with

$$\tilde{W} = \omega r - \frac{\ell + 1}{r} \tag{33}$$

and calculating

$$\tilde{V}_\pm = \tilde{W}^2 \pm \frac{\mathrm{d}}{\mathrm{d}r}\tilde{W} \tag{34}$$

leads to

$$\begin{aligned} \tilde{V}_+(r) &= \frac{\ell(\ell + 1)}{r^2} - \omega(3 + 2\ell) + \omega^2 r^2 \\ \tilde{V}_-(r) &= \frac{(\ell + 1)(\ell + 2)}{r^2} - \omega(1 + 2\ell) + \omega^2 r^2. \end{aligned} \tag{35}$$

Consequently, the model is supersymmetric and shape invariant.

4. The Bertlmann–Martin inequality

In a previous work, use has been made of the Calogero model to check the Bertlmann–Martin inequality (BMI) in the case of a potential in $D = 2$ without azimuthal symmetry [9]. We recall that the BMI connects the mean square radius of the ground state to the lowest dipole excitation energy [10]. It is derived by minimizing the dipole sum rule and then using the closure relation.

In the present model, we shall consider two types of operators, namely $r\,\mathrm{e}^{\mathrm{i}\theta}$ and $r\cos\theta$, which differ in their angular part. Whereas $\mathrm{e}^{\mathrm{i}\theta}$ connects the ground state to all states, including the radial excited states ($\Delta k \neq 0$, $\Delta n = 0$), $\cos\theta$ is more selective and has no transition from the ground state to the radial excited states. The double commutators

$$[[H, r\,\mathrm{e}^{\mathrm{i}\theta}], r\,\mathrm{e}^{-\mathrm{i}\theta}] = -4 \tag{36}$$

and

$$[[H, r\cos\theta], r\cos\theta] = -2 \tag{37}$$

lead to the two following sum rules:

$$\sum_{k,n}(E_{k,n} - E_{0,0})|\langle k, n|r\,\mathrm{e}^{\mathrm{i}\theta}|0, 0\rangle|^2 = 2 \tag{38}$$

(case I) and

$$\sum_{k,n}(E_{k,n} - E_{0,0})|\langle k, n|r\cos\theta|0, 0\rangle|^2 = 1, \tag{39}$$

(case II).

In both cases, the inequality is obtained by replacing all energy differences by the lowest one and then using closure. In the present model, according to the spectrum equation (23)

$$\langle r^2\rangle \leqslant \frac{2}{\Delta(E)} + |\langle 0, 0|r\,\mathrm{e}^{\mathrm{i}\theta}|0, 0\rangle|^2 = \frac{2}{\Delta(E)} + |\langle 0, 0|\,\mathrm{e}^{\mathrm{i}\theta}|0, 0\rangle|^2\langle r^2\rangle \tag{40}$$

and

$$\langle r^2\cos^2\theta\rangle \leqslant \frac{1}{\Delta(E)}; \qquad \langle r^2\rangle \leqslant \frac{1}{\langle\cos^2\theta\rangle}\frac{1}{E_{0,1} - E_{0,0}}, \tag{41}$$

where $\langle A\rangle$ is meant to be the ground state average of A.

Here $\Delta(E) = \inf(E_{0,1} - E_{0,0}, E_{1,0} - E_{0,0})$ in case I, and $\Delta(E) = E_{0,1} - E_{0,0}$ in case II. In equation (41) use is made of the fact that the average of the operator $r\cos(\theta)$ is zero.

According to the wavefunction equation (21), the ground state moments are given by

$$\langle r^k\rangle = \frac{\Gamma(k/2 + \ell + 1)}{\Gamma(\ell + 1)\omega^{k/2}}. \tag{42}$$

In particular the mean-square radius takes the form

$$\langle r^2\rangle = \frac{\ell + 1}{k} = \frac{1 + 2\sqrt{(1 - m)m}}{k}. \tag{43}$$

Further, we have

$$\langle \mathrm{e}^{\mathrm{i}\theta}\rangle = \mathrm{i}\left[\frac{\Gamma(2m + 1)^2}{\Gamma(2m + 3/2)\Gamma(2m + 1/2)}\right], \qquad \langle\cos^2\theta\rangle = \frac{1}{2(2m + 1)}. \tag{44}$$

For the energy differences, we have respectively

$$\begin{aligned}\Delta(E) &= 2\omega\inf\left(2, \sqrt{1 + 8m - 4m^2} - \sqrt{4m - 4m^2}\right) \qquad &\text{case I}\\ &= 2\omega\left(\sqrt{1 + 8m - 4m^2} - \sqrt{4m - 4m^2}\right) \qquad &\text{case II.}\end{aligned} \tag{45}$$

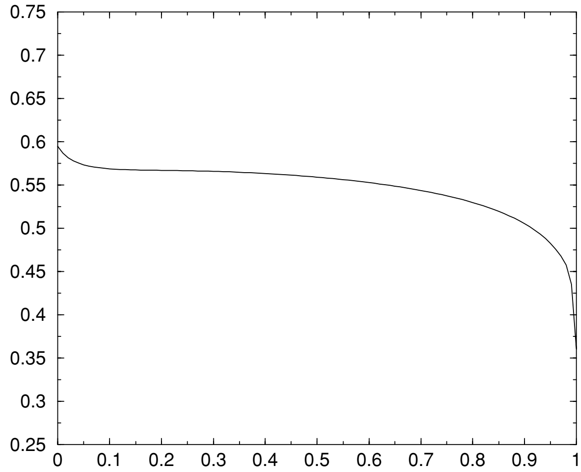

Figure 1. $m \to I_1(m)$.

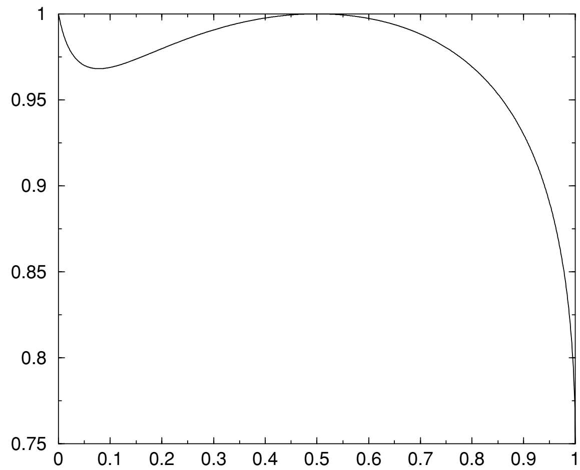

Figure 2. $m \to I_2(m)$.

The test of the Bertlmann–Martin inequality proceeds by

$$I_1 = \frac{1}{2}\langle r^2\rangle \Delta(E)\left(1 - \frac{\Gamma(2m+1)^4}{\Gamma(2m+3/2)^2\Gamma(2m+1/2)^2}\right) \leqslant 1 \tag{46}$$

in case I and

$$\begin{aligned} I_2 &= \Delta(E)\langle r^2\rangle \frac{1}{2(2m+1)} \leqslant 1 \\ &= \frac{(\sqrt{1+8m-4m^2} - \sqrt{4m-4m^2})(1+2\sqrt{(1-m)m})}{2m+1} \leqslant 1, \end{aligned} \tag{47}$$

in case II.

The curves $m \mapsto I_1, I_2(m)$ are shown in figures 1 and 2. In case I the results average around 0.55 (to be compared to unity) and deteriorate for m close to unity. In case II the Bertlmann–Martin inequality saturates for $m = 0$ and $m = 1/2$, as expected for the pure oscillator. It is known that the one-phonon excitation of the harmonic oscillator exhausts the sum rule, and thus saturates the Bertlmann–Martin inequality. As for case I we observe poorer results for $m \simeq 1$.

5. Conclusions

We have proposed a new Hamiltonian of the Calogero type. It describes two interacting particles in a harmonic mean field. The model is integrable, supersymmetric and shape invariant. For a certain domain of the coupling constant, we have drawn attention to the irregular solutions. Being square integrable, they cannot be excluded and they have a ground state energy lower than that of the regular solution.

The Bertmann–Martin inequality has been investigated, with particular emphasis being placed on the type of the operator considered. The inequality is the best saturated when the operator does not cause transitions from the ground to radially excited states. In this respect, when the coupling constant is zero, the saturation of the inequality characteristic of the harmonic oscillator is recovered.

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